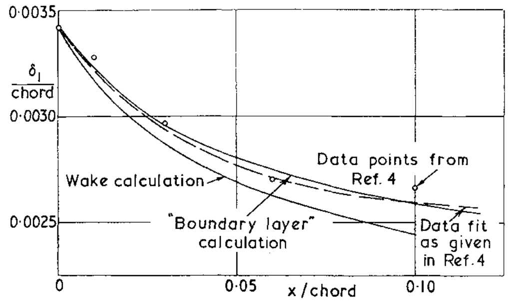

Fig. 3 Comparison with the data of Ref. 4: displacement thickness.

tion at the intersection of the two curves. We have chosen $K_1 = 0.26$ to give the best agreement between Chevray's data and calculations by the method of Ref. 1, using a length scale L equal to $0.4\ yK_1(y/y_c)^{-3/2}$ near $y = 0$—which is virtually the same as using a mixing length varying in the same way. The results for $K_1 = 0.26$ are shown in Figs. 1–2. The downstream limit of validity, the value of x at which y_c reaches 0.2δ, is marked on Fig. 1. Fig. 2 shows the shear stress profile at about this value of x. Chevray's measurements were at rather low Reynolds number, about 2000 based on momentum thickness, so that the optimum value of K_1 probably contains an inbuilt allowance for a thick viscous sublayer, though the effect of the latter on $d\delta_1/dx$ would be felt only for a few sublayer thicknesses downstream of the trailing edge.

Figure 3 shows a comparison, using $K_1 = 0.26$, with the data of Firmin[4] for an R.A.E. 101 aerofoil at zero incidence. At the trailing edge, δ_{995} is about 0.018 of the chord. The broken line in Fig. 3 is the data fit as given in Ref. 4, including points further downstream. The agreement is not as good as with Chevray's data, but Firmin did not measure shear stress profiles and the starting conditions for the wake calculation are rather uncertain (for details, see Ref. 7). The fact that the experiments were done at $M = 0.4$ and the calculations at $M = 0$ should not have affected the results appreciably. We have not yet written a program for the compressible wake but an indirect check on compressibility effects was made by doing a boundary-layer calculation, with the same pressure distribution and initial conditions as in the wake, at $M = 0.4$ and $M = 0$: the incompressible shape factor $H \equiv \delta_1/\delta_2$ (ignoring density changes), agreed to within about 0.005. This calculation revealed that the variation of displacement thickness in the "boundary layer" and wake calculations was almost identical (in fact the boundary layer values are closer to the experimental data). This is undoubtedly a coincidence; it is far from true for Chevray's constant-pressure wake, but the same may happen in other aerofoil wakes in strong favourable pressure gradient.

The simple "mixing length" fit used here is not valid once the inner wake has spread outside the inner layer of the boundary layer. In the outer layer, y_c/δ_{995} is an extra parameter, changes in the energy-diffusion function G will occur and, most important of all, the flow will no longer be self-preserving in Townsend's sense. A comparison[7] of the calculated shear stress profiles with Chevray's measurements shows that large errors accumulate for $x > 50$ cm. However, the mixing length fit should be valid far enough downstream for displacement surface calculations, and a more refined treatment must await more data on the turbulence structure of wakes. In asymmetrical wakes, separate calculations for each side will give a first approximation to δ_1. To predict the profiles we need data on the interaction between opposing shear layers.

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Correlation between Turbulent Shear Stress and Turbulent Kinetic Energy

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Nomenclature

a_1	=	constant of proportionality in relation between turbulent shear stress and turbulent kinetic energy
k	=	turbulent kinetic energy
Re	=	Reynolds number
U	=	mean velocity
u',v',w'	=	components of turbulent fluctuation velocity
ρ	=	density
τ	=	turbulent shear stress

Subscripts

j	=	jet
o	=	outer stream

Other symbol

$\langle\ \rangle$	=	time-average

Introduction

SINCE the development of the mixing length concept by Prandtl[1] in 1925, analytical investigations of turbulent flow phenomena generally have used some form of a locally dependent shear stress model. Today a number of such models are available, yet none can be applied to the analysis of a wide variety of turbulent flow problems with reasonable confidence. It is obvious that a more fundamental approach is needed. One such approach involves the use of the turbulent kinetic energy equation.

One of the first investigators to consider this approach was Nevzgljadov, cited in Ref. 2, who proposed to select the mean velocity, the mean pressure, and the turbulent kinetic energy as the independent variables in a turbulent boundary layer analysis. He further proposed a relation in which the

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turbulent shear stress is proportional to the product of the turbulent kinetic energy by the local mean velocity gradient. Dryden,[2] in discussing Nevzgljadov's work, points out that the boundary-layer studies that he reports support a more direct relation between the shear stress and the turbulent kinetic energy.

A linear relationship between turbulent shear stress and turbulent kinetic energy was used by Bradshaw et al.[3] in the analysis of turbulent boundary layer flow and by Lee and Harsha[4] in a turbulent free mixing analysis. In the latter work it was necessary to modify the linear relationship in flow regions in which the turbulent shear stress approaches zero while the turbulent kinetic energy does not. However, over the greater part of the flowfield in all of the cases considered a constant ratio of turbulent shear stress to turbulent kinetic energy was assumed to exist.

Good agreement between calculated and measured velocity profiles and shear stress profiles was achieved in both studies. However, because of the number of assumptions inherent in the use of the turbulent kinetic energy equation, this agreement is not necessarily a validation of the assumed linear relationship between turbulent shear stress and turbulent kinetic energy. A limited amount of experimental evidence for the assumption of a linear relationship was presented in Refs. 3 and 4; it is the purpose of this note to present further experimental verification of this assumption for a broad range of flow conditions.

Data Correlation

Measurements of turbulence structure for incompressible flows are available in the literature for boundary-layer flows as well as wakes and jets. The parameters of concern in this correlation are the dimensionless shear stress, defined as

$$\tau/\rho U^2 = -\langle u'v'\rangle/U^2 \tag{1}$$

and the dimensionless turbulent kinetic energy per unit mass, defined as

$$k/U^2 = (1/2U^2)[\langle u'^2\rangle + \langle v'^2\rangle + \langle w'^2\rangle] \tag{2}$$

Data for both τ and k were obtained from a wide variety of flows. In all cases both τ and k were evaluated at the same spatial position. In evaluating k, $\langle w'^2\rangle$ was assumed equal to $\langle v'^2\rangle$ if not measured.

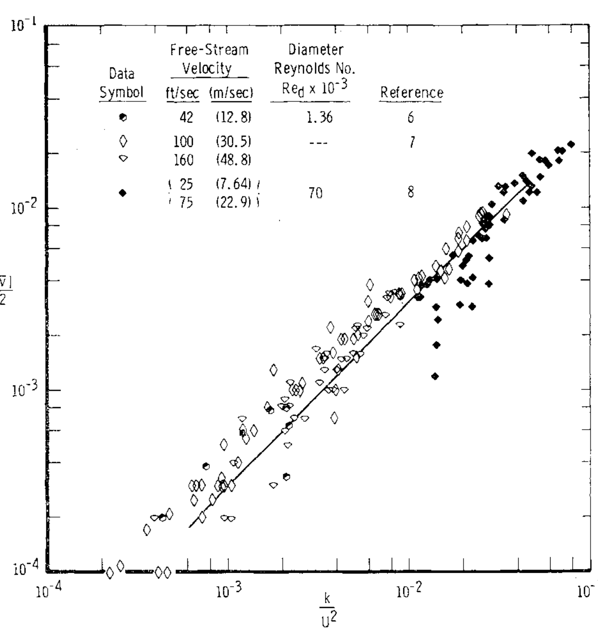

Fig. 1 Relation between turbulent shear stress and turbulent kinetic energy for wakes.

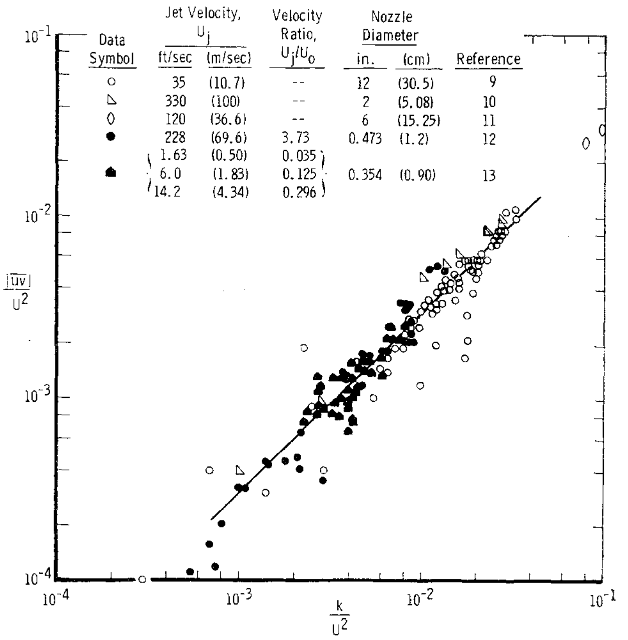

Fig. 2 Relation between turbulent shear stress and turbulent kinetic energy for circular jets.

Because of space limitations, only data relating to wakes and free jets will be presented here in detail. A discussion of the relationship between turbulent shear and turbulent kinetic energy in boundary-layer flows may be found in Bradshaw.[5]

Wakes

Three types of turbulent wake flows are represented in this section. Townsend[6] made measurements of the turbulence structure in the similarity region of a two-dimensional wake behind a cylinder of 0.0625-in. (0.159-cm) diam. Lee[7] investigated the two-dimensional wake behind the trailing edge of a symmetric airfoil with stream velocity ratios of 1.0 and 2.857. Carmody[8] investigated the axisymmetric wakes behind two discs, one 6-in. (15.25-cm) diam and the other 2-in. (5.08-cm) diam, with the freestream velocity adjusted to hold the diameter Reynolds number constant. Measured turbulent shear stress and turbulent kinetic energy data from these three experiments are shown in Fig. 1. The straight line represents the expression

$$\tau = 0.3\rho k \tag{3}$$

which can be seen to provide a good representation of the data.

Axisymmetric jets

A considerable amount of data exists for this configuration, both with and without secondary flow. The experiments of Sami,[9] Bradshaw et al.,[10] and Gibson[11] all concern circular jets exhausting into quiescent air, while Curtet and Ricou[12] and Zawacki and Weinstein[13] both considered ducted coaxial jets. In the Curtet and Ricou experiments, the outer duct created a small pressure gradient effect; the outer duct used in the experiments of Zawacki and Weinstein had no appreciable effect on the pressure field. The shear stress and kinetic energy data from these experiments are shown in Fig. 2; the line represents Eq. (3), which can again be seen to provide a reasonably good correlation.

Summary of the data correlation

A range plot of the data considered in this study is shown in Fig. 3 in the form of a conventional bar graph. Some of the data included in constructing Fig. 3 (for boundary

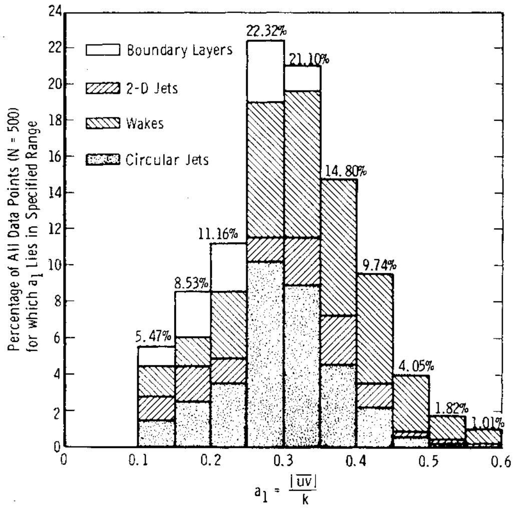

Fig. 3 Distribution of the observed values of the parameter a_1 for the data surveyed.

layers and two-dimensional jets) have not been separately plotted in this note. It can be seen that 70% of the data fall in the range $0.2 < a_1 < 0.4$, and that $a_1 = 0.3$ is a reasonably good value judging from the approximately 500 data points examined. In regard to this correlation it should be pointed out that the parameter a_1 must approach zero at the centerline of a free mixing flow, since the shear stress at the centerline must be zero by reason of flow symmetry, while experimental evidence indicates that the turbulent kinetic energy remains nonzero. The value of the parameter a_1 then must increase from zero to its nominal value over a portion of the mixing region. The experiments of Sami[9] are sufficiently detailed for this variation to be investigated, and this limited evidence indicates that the portion of the mixing region over which this variation occurs is small. A similar variation is also seen to occur in boundary-layer flows (see Bradshaw[5]). This variation in the value of the parameter a_1 may explain the slight biasing toward the smaller values of a_1 apparent in Fig. 3.

Conclusions

Based on a study of a substantial amount of turbulent shear stress and kinetic energy data, the existence of a linear relationship between turbulent shear stress and kinetic energy is reasonably well supported over a wide range of experimental conditions in incompressible flow. In the regions of flow where a constant ratio between turbulent shear and turbulent kinetic energy can be expected to exist, a reasonable value for the constant of proportionality may be taken to be 0.3. There is not yet sufficient evidence available for the variation of this constant of proportionality to be modelled in other flow regions, such as those in which the turbulent shear stress approaches zero while the turbulent kinetic energy does not.

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Exponential Kernel Approximation in Radiative Energy Transfer within a Hydrogen Plasma

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Nomenclature

e_ω = Planck's function
E_2 = second exponential integral function
F_2 = transmission function, Eq. (1), $cm^{-1}atm^{-1}$
L = plate spacing, cm
Ne = electron density, cm^{-3}
P = total pressure, atm
P_H = partial pressure of atomic hydrogen, atm
Q = heat source per unit volume
q_R = radiative heat flux
T = temperature, °K
T_1 = boundary temperature, °K
T_c = centerline temperature, °K
u = pressure path length, $u = P_H y$, cm-atm
u_o = total pressure path length, $u_o = P_H L$, cm-atm
y = distance measured from lower boundary
κ_ω = spectral absorption coefficient, cm^{-1}
λ = thermal conductivity
σ = Stefan-Boltzmann constant
ω = wave number, cm^{-1}

Introduction

THE object of this Note is to investigate the influence of the exponential kernel approximation on radiative energy transfer within high-temperature gases. In order to

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